JOURNAL OF APPROXIMATION THEORY 5, ix–xiii (1972)

Homage to Walsh*

MORRIS MARDEN

Department of Mathematics, University of Wisconsin-Milwaukee
Milwaukee, Wisconsin 53201

During the week of October 25, 1970 over 120 Mathematicians—friends, former students and colleagues—gathered to honor Joseph Leonard Walsh. They came from near and far, from the U.S.A. and abroad, to help celebrate the seventy-fifth birthday anniversary of this man, who has played a leading role in American mathematics for almost fifty years. This conference took place at the University of Maryland, not far from the scene of his childhood.

Professor Walsh was born in Washington, D.C. on September 21, 1895. Between 1908 and 1912 he attended the Baltimore Polytechnic Institute from which he entered Columbia University. After a year he transferred to Harvard with which he thus began an association that lasted half a century. His undergraduate record was outstanding, for in 1916, along with an S.B. degree summa cum laude, Harvard awarded him a Sheldon Travelling Fellowship. This allowed him to pursue mathematical studies at the University of Chicago and at the University of Wisconsin, culminating in a Master's degree granted by the latter institution in 1917. On returning to Harvard, he came again under the scientific influence of Maxime Bôcher, but his progress towards a Ph.D. was interrupted by World War I and his enlistment in the Navy.

During the war, Bôcher had died so that, on returning to Harvard, Walsh chose to complete his thesis work under the guidance of another member of the Harvard faculty, George D. Birkhoff.

Walsh's thesis was entitled "On the roots of the jacobian of two binary forms". The purpose was to generalize the following theorem of Bôcher: If f and g are two binary forms of the same degree and if all the zeros of f lie in a circular region A whereas all the zeros of g lie in a circular region B with $B \cap A = \varnothing$, then all the zeros of the jacobian of f and g lie in $A \cup B$. To carry out his purpose, Walsh employed mainly geometric and physical

* Dedicated to Professor J. L. Walsh on the occasion of his seventy-fifth birthday anniversary.

methods. The latter were based upon Bôcher's interpretation of the zeros of the jacobian as equilibrium positions in the field of force due to positive particles at the zeros of f and negative particles at the zeros of g, the attraction being according to the inverse distance law. While accomplishing this purpose, Walsh obtained a large number of interesting results of which the most beautiful were the following two:

(1) If f, a polynomial of degree n, has n_1 zeros in the disk $| z - c_1 | \leqslant r_1$ and the remaining $n_2 = n - n_1$ zeros in the disk $| z - c_2 | \leqslant r_2$, then any critical point of f not in either of these disks lies in a third disk

$$| nz - (n_2 c_1 + n_1 c_2) | \leqslant n_2 r_1 + n_1 r_2 .$$

(2) If the points z_1, z_2, z_3 have, respectively, as loci the circular regions C_1, C_2, C_3, then the point z_4 defined by the cross ratio $(z_1, z_2, z_3, z_4) = \text{const.}$ has as locus also a circular region C_4.

On receiving his Ph.D. from Harvard in 1920, Walsh was awarded a second Sheldon Travelling Fellowship. This time he went abroad for a year of study at the University of Paris where he consulted with Paul Montel; he then returned to a teaching post at Harvard. During the next few years, associating with G. D. Birkhoff and O. D. Kellogg, he developed an active interest in orthogonal functions. Among other discoveries, he announced in 1923 the construction of a complete orthonormal system that is now known as the Walsh system and that has been found highly useful to engineers as well as to mathematicians. During 1925–6 he was a recipient of an International Research Board Fellowship which permitted him to spend a year at the University of Munich in association with C. Carathéodory. As might be expected, these trips abroad helped further broaden his lines of mathematical activity as is indicated by the variety of topics covered in his subsequent papers.

Of these topics, one general topic is interpolation and approximation. We recall his theorem that a function which is analytic in a bounded Jordan region R and continuous in the closure $\bar{R}$ of R can be uniformly approximated in $\bar{R}$, arbitrarily closely, by polynomials. Another of his theorems states that if a function f is uniformly approximable as closely as desired by polynomials, on a closed bounded set S, then such approximating polynomials can be chosen as also interpolate to f on any given, finite subset of S. In his work on approximation he was aided by his discovery of a new canonical conformal map for multiply connected regions. These and many other of his results on approximation (including degree of convergence) may be found in his now classical monograph "Interpolation and Approximation by Rational Functions in the Complex Domain" (1935; fifth edition, 1969).

In addition, there are several general topics which grew out of his earlier studies on the critical points of polynomials. We mention first the subjects of critical points of harmonic functions and harmonic measures, for which he obtained various interesting results, some being analogous to earlier results on polynomials. In particular, one recalls his study of Green's function $G(x, y)$, with pole at infinity, for an unbounded region R having a bounded boundary B of a very general type.

Having developed an integral representation for G in R, in terms of this function on B, he proved the striking Lucas-type result that all critical points of G in R lie in the convex hull of B. He found also numerous results regarding the geometry of the level curves of $G(x, y)$. These developments are described in his monograph, "The Location of Critical Points of Analytic and Harmonic Functions", 1950.

Another topic is that of extremal polynomials and infrapolynomials. This is a topic which was thoroughly explored in his many papers written both independently and in collaboration with Professors Fekete, Motzkin, Shisha and Zedek. It is a topic to which he was led by his earlier interests both in critical points of polynomials and in approximation theory.

Still more recent is Walsh's work on spline theory. Some of his contributions to this topic are described in the monograph "The Theory of Splines and Their Applications", 1967, jointly with J. H. Ahlberg and E. N. Nilson.

To return to the account of Walsh's career, we note his rise at Harvard through the various ranks from instructorship to full professorship which he attained in 1935. In 1946, Harvard recognized his accomplishments by naming him Perkins Professor of Mathematics. This was a very high honor since the chair had been earlier occupied by such Harvard notables as William F. Osgood and George D. Birkhoff. Walsh held this chair until his retirement from Harvard in 1966. But it is characteristic of him that he began his present, second active career, at the University of Maryland, half a year ahead of becoming an emeritus professor at Harvard.

His qualities were recognized not only at Harvard but also among academic and other circles throughout the country and abroad. Evidence of this are his election in 1936 as member of the National Academy of Sciences, in 1937 as vice-president of the American Mathematical Society and in 1949 as president of the Society.

Walsh had joined the Navy in 1918 as an ensign, serving in troop transports in the North Atlantic. He was promoted to lieutenant before the end of World War I. During World War II he served as a navigator on an aircraft carrier in the rank of lieutenant commander. Before his retirement from the Naval Reserve in 1955, he attained the rank of captain.

To the above information on Walsh, which can be gleaned from biographical listings, old catalogues of the Harvard Mathematics Department,

etc., I should like to add a few notes from my own recollections. I made Walsh's acquaintance sometime during my freshman or sophmore year at Harvard (1921–3). I remember particularly the blue Navy pea jacket which he wore throughout the winter months. My close association with him began in my junior year when he choose me as his assistant (problem reader) in his first offering of a course in theoretical mechanics. I believe, however, the only formal course that I took under him was in my senior year and it was a course on real variables. I still recall the deep musical sound of his voice as he lectured on this subject with meticulous care.

When I entered the Harvard Graduate School in 1925, my interests were along the lines of differential equations and so I started a research program under Professor George D. Birkhoff, together with a part-time teaching appointment. To my dismay I learned in the Spring of 1926, that Birkhoff expected to be away from Harvard on a trip around the world during all of the academic year 1926–7 and so I would be left without a thesis advisor. Fortunately for me, Walsh was returning from a year's leave of absence. When I visited him in the Fall of 1926, he spoke with such enthusiasm about the open problems in his field that I asked to have a try at them. Thus at my age of 21 and his age of 31 I became his first Ph.D. student and, in June 1928, his first Ph.D. alumnus.

We were all so young and energetic in those days. I remember the long hikes which Walsh and I took along with other members of the Harvard Mathematics Club. One was up and down across the Blue Hills near Boston and another was along the North Shore from Winthrop (my home town) to Salem—a distance of about 25 miles.

Walsh seemed to have been especially fond of outings. However, in 1926 one of them ended in a serious mishap. As the story went, he climbed a tree on a dare, but then lost his balance and broke his left hand in the resulting fall. Upon his discharge from the hospital, we had to transfer our conferences to his quarters in Perkins Hall, a graduate student dormitory where he was serving as proctor. There, as we discussed the progress of my research, he sat in his easy chair flexing his fingers over a rubber ball, striving to restore the muscles of his hand. His persistence paid off: he achieved a nearly complete recovery.

In later years, after moving to Wisconsin, I was invited occasionally along with others to his summer place at Randolph, New Hampshire. His house had a detached study which was situated with a full view of the White Mountains. He knew these mountains quite intimately and was very adept at climbing them. In fact, he ascended at such a speed that we, younger men, had difficulty in keeping up with him.

All in all, Walsh has been a marvelous person to have studied under, and to have had as a friend over these years. I am sure that the many men

and women who subsequently were his students share these sentiments with me. To us it has been an inspiration to see him remain vigorous and productive. We all look forward confidently to his continuation of an active life and to our celebration with him of this eightieth birthday anniversary in 1975.

March 5, 1971
University of Wisconsin-Milwaukee